INVERSE THERMOELASTIC PROBLEM IN AN ANISOTROPIC CYLINDER

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ABSTRACT

In this paper an attempt has been made to solve the inverse problem of thermoelasticity in order to determine the heating temperature, temperature distribution and thermal stresses on the outer surface of an anisotropic cylinder defined as with the help of integral transform technique.

Keywords: *Inverse thermoelastic problem, thermal deflection, circular plate.*

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1. Introduction

The inverse thermoelastic problem consists of determination of the temperature of the heating medium, the heat flux on the boundary surfaces of the solids when the conditions of the displacement and stresses are known at some points of the solid under consideration. The inverse problem is very important in view of its relevance to various industrial machines subjected to heating such as main shaft of lathe and turbines roll of rolling mills, to analysis of experimental data and measurement of aerodynamic heating. Grysa and Cialkawski [5] and Grysa and Kozlowski [6] one dimensional transient thermoelastic problems derived from the heating temperature and the heat flux on the surface of an isotropic infinite slab are investigated. Deshmukh and Wankhede [3] are investigated the temperature, displacement and stress functions infinite isotropic hollow cylinders of small thickness. Direct problems on anisotropic bodies are considered Nowacki W. [7, 8], Nowinski [9], Avtar Sing [1] and D. Rama Murthy [2].
Recently Deshmukh [4] studied the inverse problem of an anisotropic body and determined the thermal stresses. Hence an attempt to solve the inverse thermoelastic problem in an anisotropic cylinder is consider to determine the heating temperature, temperature distribution and thermal stresses on the outer surface of the cylinder defined as with the help of integral transform technique. The results are illustrated in the form of series.

2. Statement of the Problem

Consider an infinite circular cylinder. The surfaces $r=0$ and $r=a$ are bounded by the planes $\theta=0$ to $\theta=\theta_0$. On the planes $\theta=0$ and $\theta=\theta_0$, temperature is kept zero. Known temperature is taken on the surface $r=a$, where $0<a<b$. The problem is treated as one inverse quasi-static thermoelasticity i.e. the variation occurs with respect to time occurs only so far as the temperature's concerned while the stresses and displacement with respect to time are neglected hence, in the field equations energy equation only contains what is known as the term relating to the velocity of heat propagation. A materials with cylindrical anisotropic are taken into consideration.

Basic Equation

The heat transport for cylindrically aelotropic materials is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right]+\frac{p^2}{r^2}\frac{\partial^2 T}{\partial \theta^2}=\frac{p^2}{\beta^2}\frac{\partial T}{\partial t} \tag{2.1}$$

Where $p^2=\frac{k_2}{k_1}$ and $\beta^2=\frac{k_2}{\rho_c}$

k_1, k_2 Are thermal conductivity in r and θ directions respectively. Equation (7.4.1) can be written as

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{p^2}{r^2}\frac{\partial^2 T}{\partial \theta^2}=\frac{p^2}{\beta^2}\frac{\partial T}{\partial t} \tag{2.2}$$

$0\le r\le a$, $0\le\theta\le\theta_0$, $t>0$

with the initial condition

$$T(r,\theta,0)=0 \tag{2.3}$$

and the boundary conditions

$$T(r,0,t)=0\,, \tag{2.4}$$

$$\left(\frac{\partial T}{\partial \theta}\right)_{\theta=\theta_0}=0\,, \tag{2.5}$$

$$\left(\frac{\partial T}{\partial r}\right)_{r=a} = f(\theta, t) \text{ (unknown)} \qquad (2.6)$$

and the interior condition

$$T(\xi, \theta, t) = g(\theta, t) \text{ (known)} \qquad (2.7)$$

The equations (2.1) to (2.7) constitute the mathematical formulation of heat conduction under consideration.

3. Solution of the Problem: Determination of the Temperature and Unknown Function

Applying the finite Fourier sine transform to the equations (2.2), (2.3), (2.6), (2.7) and using the conditions (2.4), (2.5) and then finally taking Laplace transform using the condition (2.3) one gets

$$\frac{d^2\overline{T}_s}{dr^2} + \frac{1}{r}\frac{d\overline{T}_s}{dr} - \left(q^2 + \frac{\gamma^2}{r^2}\right)\overline{T}_s = 0 \qquad (3.1)$$

where $q^2 = \frac{p^2 s}{\beta^2}$, $\gamma^2 = p^2\beta_m^2$

$$\overline{T}_s(r, \beta_m, 0) = 0 \qquad (3.2)$$

$$\left(\frac{d\overline{T}_s}{dr}\right)_{r=a} = \bar{f}_s(\beta_m, s) \qquad (3.3)$$

and $\overline{T}_s(\xi, \beta_m, s) = \bar{g}_s(\beta_m, s)$ (3.4)

where $\overline{T}_s$ is Laplace transform of T_s and s is its parameter.

The solution of equation (7.5.9) are

$$\overline{T}_s = c_1 I_\gamma(qr) + c_2 K_\gamma(qr) \qquad (3.5)$$

where $I_\gamma(qr)$ and $K_\gamma(qr)$ are the modified Bessel's functions of order γ and c_1 and c_2 are the constants. As r tends to zero $K_\gamma(qr)$ tends to infinity but $\overline{T}_s$ are finite therefore the constant c_2 must be zero. Hence one gets

$$\overline{T}_s = c_1 I_\gamma(qr) \qquad (3.6)$$

On applying the condition one gets

$$\overline{T}_s = \bar{g}_s(\beta_m, s)\frac{I_\gamma(qr)}{I_\gamma(q\xi)} \qquad (3.7)$$

$$\bar{f}_s(\beta_m, s) = \bar{g}_s(\beta_m, s)\, q\frac{I'_\gamma(qa)}{I_\gamma(q\xi)} \qquad (3.8)$$

Taking inverse Laplace transform to equations (3.1) and (3.8) and taking the inverse sine transform to the resultants one gets the expression of temperature $T(r, \theta, t)$ and unknown function $f(\theta, t)$ respectively as

$$T(r, \theta, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{\lambda n} e^{-\beta^2\lambda_n^2 t/p^2} \sin(\beta_m\theta)\, J_\gamma(\lambda_n r) \qquad (3.9)$$

$$f(\theta, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \lambda_n A_{\lambda n} \sin(\beta_m\theta)\, J'_\gamma(\lambda_n a)\, e^{-\beta^2\lambda_n^2 t/p^2} \qquad (3.10)$$

$$\text{where } A_{\lambda n} = -\frac{2\beta^2}{p^2\xi}\sqrt{\frac{2}{\theta_0}}\frac{\lambda_n}{J'_\gamma(\lambda_n\xi)}\int_0^t g_s(\beta_m, u)\, e^{\beta^2\lambda_n^2 u/p^2}\, du$$

m, n are the positive integers and $\lambda_1, \lambda_2, \lambda_3, \Lambda, \lambda_n$ are the roots of the transudental equations

$$J_\gamma(\lambda_n\xi) = 0 \qquad (3.11)$$

4. Solution of the Associated Problem of Thermal Stresses

The stresses in terms of the stress function $F(r, \theta, t)$ are given by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2} \qquad (4.1)$$

$$\sigma_{\theta\theta} = \frac{\partial^2 F}{\partial r^2} \qquad (4.2)$$

$$\sigma_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta}\left(\frac{F}{r}\right) \qquad (4.3)$$

since the problem is that of quasi-static one the equations of equilibriums in terms of stresses are satisfied by the above expressions. All the compatibility equations except one are satisfied. The only equation to be satisfied is given by

$$\frac{\partial^2}{\partial r \partial \theta}(r e_{r\theta}) = \frac{\partial}{\partial r}\left[r^2 \frac{\partial e_{\theta\theta}}{\partial r}\right] + \frac{\partial^2 e_{rr}}{\partial \theta^2} - r\frac{\partial e_{rr}}{\partial r} \qquad (4.4)$$

for cylindrically aelotropic materials the stress-strain relations are given by

$$e_{rr} = a_{11}\sigma_{rr} + a_{12}\sigma_{\theta\theta} + a_1 T \qquad (4.5)$$

$$e_{\theta\theta} = a_{12}\sigma_{rr} + a_{22}\sigma_{\theta\theta} + a_2 T \qquad (4.6)$$

$$e_{r\theta} = a_{66}\sigma_{r\theta} \qquad (4.7)$$

substituting (4.1), (4.2), (4.3), (4.5), (4.6) and (4.7) in (4.4) one gets

$$\left[r^4 \frac{\partial^4}{\partial r^4} + 2r^3 \frac{\partial^3}{\partial r^3} - \beta_1 r^2 \frac{\partial^2}{\partial r^2} - \beta_2 r \frac{\partial^3}{\partial r \partial \theta^2} + \beta_1 r \frac{\partial}{\partial r} + \beta_2 r^2 \frac{\partial^4}{\partial r^2 \partial \theta^2} + (2\beta_1 + \beta_2)\frac{\partial^2}{\partial \theta^2} + \beta_1 \frac{\partial^4}{\partial \theta^4}\right] F = -\gamma_1 \left[r^4 \frac{\partial^2}{\partial r^2} + (2-\mu) r^3 \frac{\partial}{\partial r} + \mu r^2 \frac{\partial^2}{\partial \theta^2}\right] T \qquad (4.8)$$

where $\beta_1 = \frac{a_{11}}{a_{22}}$, $\beta_2 = \frac{(2a_{12} + a_{66})}{a_{22}}$, $\mu = \frac{a_1}{a_2}$, $\gamma_1 = \frac{a_2}{a_{22}}$,

a_1, a_2 are the coefficients of thermal expansions in r and θ directions respectively on putting the value of T from equation (3.9) in equation (4.8) and noting that

$$J_\gamma(\lambda_n r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \overline{| k+\gamma+1}} \left(\frac{\lambda_n}{2}\right)^{2k+\gamma} r^{2k+\gamma} \qquad (4.9)$$

one gets

$$\left[r^4 \frac{\partial^4}{\partial r^4} + 2r^3 \frac{\partial^3}{\partial r^3} - \beta_1 r^2 \frac{\partial^2}{\partial r^2} - \beta_2 r \frac{\partial^3}{\partial r \partial \theta^2} + \beta_1 r \frac{\partial}{\partial r} + \beta_2 r^2 \frac{\partial^4}{\partial r^2 \partial \theta^2} + (2\beta_1 + \beta_2)\frac{\partial^2}{\partial \theta^2} + \beta_1 \frac{\partial^4}{\partial \theta^4}\right] F = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} M_{km} r^{2k+\gamma+2} A_{\lambda n} \sin(\beta_m \theta)\, e^{-\beta^2 \lambda_n^2 t / p^2} \qquad (4.10)$$

The solution of equation (4.10) is obtained by assuming

$$F(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_m(r) A_{\lambda n} \sin(\beta_m \theta) e^{-\beta^2 \lambda_n^2 t / p^2} \qquad (4.11)$$

On putting (4.11) in (4.10) one gets

$$F(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[C_{m1} r^{1+\rho_1} + C_{m2} r^{1+\rho_2} + \sum_{k=0}^{\infty} \frac{M_{km}}{B_k} r^{2k+\gamma+2}\right] \times A_{\lambda n} \sin(\beta_m \theta) e^{-\beta^2 \lambda_n^2 t / p^2} \qquad (4.12)$$

where C_{m1} and C_{m2} are the arbitrary constants and ρ_1 and ρ_2 are roots of

$$\rho_i^4 - \rho_i^2 (1 + \beta_1 + \beta_2 \beta_m^2) + \beta_1 (\beta_m^2 - 1)^2 = 0, \quad i = 1, 2, \Lambda \qquad (4.13)$$

and

$$M_{km} = -\gamma_1 [(2k+\gamma)(2k+\gamma-1) + (2-\mu)(2k+\gamma) - \mu \beta_m^2] \times \frac{(-1)^k}{k! \overline{| k+\gamma+1}} \left(\frac{\lambda_n}{2}\right)^{2k+\gamma} \qquad (4.14)$$

Substituting (4.12) in equation (4.1) and (4.3) one gets

$$\sigma_{rr} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{C_{m1}r^{\rho_1-1}\left[1+\rho_1-\beta_m^2\right]+C_{m2}r^{\rho_2-1}\left[1+\rho_2-\beta_m^2\right] + \sum_{k=0}^{\infty}\frac{M_{km}}{B_k}(2k+\gamma+2-\beta_m^2)r^{2k+\gamma}\Big\} A_{\lambda n}\sin(\beta_m\theta)e^{-\beta^2\lambda_n^2 t/p^2} \tag{4.15}$$

$$\sigma_{r\theta} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -\left\{\rho_1 C_{m1}r^{\rho_1-1}+\rho_2 C_{m2}r^{\rho_2-1}+\sum_{k=0}^{\infty}\frac{M_{km}}{B_k}(2k+\gamma+1)r^{2k+\gamma}\right\} \times\beta_m A_{\lambda n}\cos(\beta_m\theta)e^{-\beta^2\lambda_n^2 t/p^2} \tag{4.16}$$

where

$$B_k = (2k+\gamma+2)^4-4(2k+\gamma+2)^3+[5-(\beta_1+\beta_2\beta_m^2)](2k+\gamma+2)^2 - 2[1-(\beta_1+\beta_2\beta_m^2)]\,(2k+\gamma+2)+[-(2\beta_1+\beta_2)\beta_m^2+\beta_1\beta_m^4] \tag{4.17}$$

The constants C_{m1} and C_{m2} are determined using the boundary conditions which is taken as

$$\sigma_{rr}=\sigma_{r\theta}=0 \text{ at } r=a$$

On putting $r=a$ in (4.15) and (4.16) one gets

$$C_{m1} = -\sum_{k=0}^{\infty}\frac{M_{km}}{B_k}\frac{a^{2k+\gamma-\rho_1+1}}{(\rho_1-\rho_2)(1-\beta_m^2)}\{(2k+\gamma+1)(1+\rho_2-\beta_m^2)+\rho_2[\beta_m^2-(2k+\gamma+2)]\} \tag{4.18}$$

$$C_{m2} = \sum_{k=0}^{\infty}\frac{M_{km}}{B_k}\frac{a^{2k+\gamma-\rho_2+1}}{(\rho_1-\rho_2)(1-\beta_m^2)}\times\{(2k+\gamma+1)(1+\rho_1-\beta_m^2)+\rho_1[\beta_m^2-(2k+\gamma+2)]\} \tag{4.19}$$

5. Special Case

Setting $g(\theta,t)=\theta(\theta-\theta_0)^2 t$ (5.1)

Applying finite Fourier sine transformed defined in equation (5.1) one gets

$$g_s(\beta_m,t) = 2t\sqrt{\frac{2}{\theta_0}}\frac{1}{\beta_m^4}[\beta_m\theta_0(2+\cos\beta_m\theta_0)-3\sin(\beta_m\theta_0)] \tag{5.2}$$

Substituting equation (5.2) in equations (3.9), (3.10), (4.15) and (4.16) one obtains

$$T(r,\theta,t) = -\frac{8p^2}{\xi\beta^2\theta_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(\beta_m\theta_0)}{\lambda_n^3\beta_m^4}\frac{J_\gamma(\lambda_n r)}{J'_\gamma(\lambda_n r)}[\beta_m\theta_0(2+\cos\beta_m\theta_0)-3\sin(\beta_m\theta_0)] \times\left[\frac{\beta^2\lambda_n^2 t}{p^2}+(e^{-\beta^2\lambda_n^2 t/p^2}-1)\right] \tag{5.3}$$

$$f(\theta,t) = -\frac{8p^2}{\xi\beta^2\theta_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(\beta_m\theta_0)}{\lambda_n^2\beta_m^4}\frac{J'_\gamma(\lambda_n a)}{J'_\gamma(\lambda_n\xi)} \times[\beta_m\theta_0(2+\cos\beta_m\theta_0)-3\sin(\beta_m\theta_0)]\left[\frac{\beta^2\lambda_n^2 t}{p^2}+(e^{-\beta^2\lambda_n^2 t/p^2}-1)\right] \tag{5.4}$$

$$\sigma_{rr} = -\frac{8p^2}{\xi\beta^2\theta_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(\beta_m\theta_0)}{\lambda_n^3\beta_m^4}\frac{[\beta_m\theta_0(2+\cos\beta_m\theta_0)-3\sin(\beta_m\theta_0)]}{J'_\gamma(\lambda_n\xi)} \times\Big\{C_{m1}r^{\rho_1-1}(1+\rho_1-\beta_m^2)+C_{m2}r^{\rho_2-1}(1+\rho_2-\beta_m^2) + \sum_{k=0}^{\infty}\frac{M_{km}}{B_k}(2k+\gamma+2-\beta_m^2)\,r^{2k+\gamma}\Big\}\left[\frac{\beta^2\lambda_n^2 t}{p^2}+(e^{-\beta^2\lambda_n^2 t/p^2}-1)\right] \tag{5.5}$$

$$\sigma_{r\theta}=\frac{8p^2}{\xi\beta^2\theta_0}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\cos(\beta_m\theta)}{\lambda_n^3\beta_m^3}\frac{[\beta_m\theta_0(2+\cos\beta_m\theta_0)-3\sin(\beta_m\theta_0)]}{J'_\gamma(\lambda_n\xi)}$$
$$\times\left\{\rho_1 C_{m1}r^{\rho_1-1}+\rho_2 C_{m2}r^{\rho_2-1}+\sum_{k=0}^{\infty}\frac{M_{km}}{B_k}(2k+\gamma+1)\,r^{2k+\gamma}\right\} \tag{5.6}$$
$$\times\left[\frac{\beta^2\lambda_n^2 t}{p^2}+(e^{-\beta^2\lambda_n^2 t/p^2}-1)\right]$$

Concluding Remarks

In this paper we studied the inverse thermoelastic problem in an anisotropic cylinder. The analytical expressions for heating temperature, temperature distribution and thermal stresses are determined on the outer surface of the cylinder, defined as $0\le r\le a,\ 0\le\theta\le\theta_0$ with the help of integral transform technique. The results are illustrated in the form of series.

Any particular case of special interest can be derived by assigning suitable values to the parameters and functions are valid in the expressions (3.9), (3.10), (4.15) and (4.16).

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