ACHARYA POLYNOMIAL OF CACTUS CHAIN GRAPH

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ABSTRACT

For connected graph G, of order n, the ***Acharya Polynomial*** *is defined as* $AP(G,\lambda) = \sum_{\substack{1\le d\le n-1 \\ 1\le k\le p}} \mu(d,G)\cdot\lambda^k$ *, where* $\mu(d,G)$ *denotes pair of vertices of degree d at distance k and p is diam(G). In the present paper we compute Acharya polynomial of some chain graphs known as cactus graphs.*

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1. Introduction

The graphs in this paper are finite and connected. Let *G* be a simple connected graph with vertex set *V* and edge set *E*. The degree of a vertex *v* in *G* is the number of edges incident to vertex *v* . A vertex of degree 1 is called terminal vertex. The distance between two vertices u and v of *V* is the length of the shortest path between u and v and is denoted by $d(u,v)$.[1]

The Wiener index was first index used by Harold Wiener to determining the boiling point of paraffin. Since then, the index has been used to build a correlation model between the chemical structures of various chemical compounds. Wiener index is the most celebrated topological index that identifies the characteristics chemical compounds. The following are some distance based polynomials and their respective indices.

Definition 1.1: [2] The ***Wiener index*** is a graph invariant based on distance in graphs. It is denoted by *W*(*G*) and defined as sum of distances of all pair of vertices in *G*:

$$W(G) = \sum_{(u,v)\in V(G} d(u,v)$$

Definition 1.2: [3] The ***Hosoya polynomial*** of graph is a polynomial introduced by Hosoya in 1988. Hosoya polynomial (also called Wiener polynomial) of G is defined as

$$H(G,\lambda) = \sum_{k\ge1} d(G,k)\lambda^k$$

Where *d(G,k)* is the number of pair of vertices of *G* that at a distance *k* and λ is a parameter.

It is clear that, $W(G) = \frac{d}{d\lambda}H(G,\lambda)$ at $\lambda = 1$.

Definition 1.3: [4] The Terminal Wiener index is denoted by *TW*(*G*) and defined as sum of distances between all pair of terminal vertices in G.

$$TW(G) = \sum_{1<i<j<k} d(v_i, v_j / G)$$

Definition 1.4: [5] The Terminal Hosoya Polynomial of graph G is defined as,

$$TH(G,\lambda) = \sum_{k\ge1} d_T(G,k)\lambda^k$$

And $TW(G) = \frac{d}{d\lambda}TH(G,\lambda)$ at $\lambda = 1$..

Definition 1.5: [6] Let *G* be a connected graph of order *n* and degree *d*, the ***Acharya Index*** $AI_\lambda(G)$ of a graph *G* as the sum of the distance between all pair of degree *d* vertices, denote as

$$AI(G) = \sum_{\substack{1\le d\le n-1 \\ 1\le k\le p}} \mu(d,k)\cdot k$$

where $\mu(d,k)$ denotes pair of vertices of degree *d* at distance *k*, *p*=*diam*(*G*).

Definition 1.6:[7] Let *G* be a connected graph of order *n* and degree *d*, the ***Acharya***

Polynomial AP(G, λ***)*** of a graph G is defined as $\sum_{\substack{1\le d\le n-1\\ 1\le k\le p}} \mu(d,k)\cdot\lambda^k$, where $\mu(d,G)$ denotes pair of vertices of degree d at distance k and p *is diam*(G).

$$AI(G)=\frac{d}{d\lambda}AP(G,\lambda) \text{ at } \lambda=1$$

In this paper we consider a class of simple linear polymers called cactus chains. A cactus graph is a connected graph in which no edge lies in more than one cycle. thus each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size m , the cactus is m - uniform. A triangular cactus is a graph whose blocks are triangles, *i.e*., a 3 -uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus. By replacing triangles by cycles of length 4 It is called square cacti where every block is C_4 . We call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbours. If their cut-vertices are adjacent, we say that such a square is an ortho-square, if the cut-vertices are not adjacent, we call the square a parasquare [8-11].In the definition of triangular cactus ,if we replace the triangles by cycles of length 6 such cacti is called hexagonal cacti.The pattern how the cycles are connected, if the cut vertices are adjacent then such cacti is called ortho hexagonal or if they are not adjacent, then it is called para-chain cacti. Thereare 2 types para-chain hexagonal cactus are there as shown in the Figure 4 and 5.

Types of cactus graphs studied in this paper

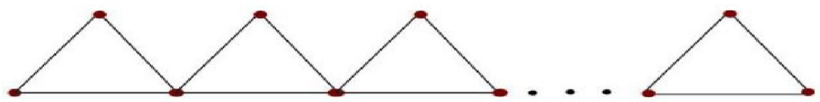

Figure 1.Chain triangular graph T_n

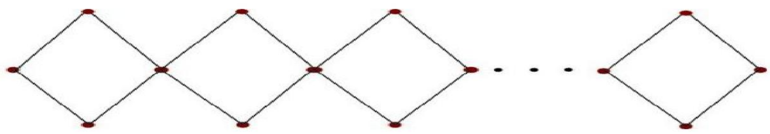

Figure 2.Chain Para-chain square cactus graph Q_n

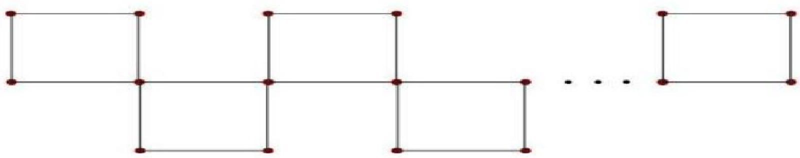

Figure 3.Chain Ortho-chain square cactus graph O_n

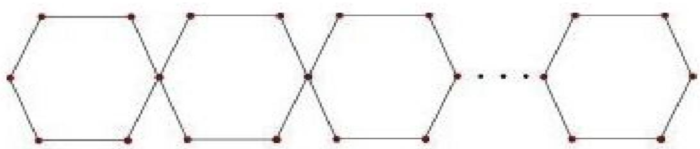

Figure 4. Para-chain hexagonal cactus graph L_n.

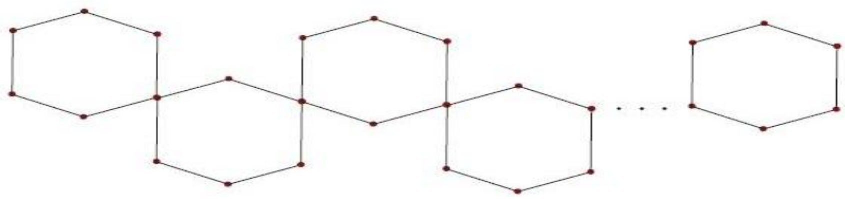

Figure 5. Meta-chain graph M_n.

Theorem 1: The Acharya polynomial of the chain cactus $T_n(n \geq 2)$ is

$$AP(G,\lambda) = n\lambda + 2\sum_{i=2}^{n-1}(n-i+1)\lambda^i + 4\lambda^n$$

$$AI(G) = \frac{n^3+3n^2+5n}{3}$$

Proof: Let x and y are any two vertices of $T_n(n \geq 2)$, then the cactus graph $T_n(n \geq 2)$ is biregular graph with degrees of vertices either 2 or 4. To find the Acharya polynomial we have find the distances between the vertices with same degree. Let the distance between the vertices be $d(x,y)=i$,then we have following distances and the number of pair of vertices with these distances.

i) There are 2 pair of degree 2 vertices and *n*-2 pair degree 4 vertices at a distance 1.Therefore we have $[2+(n-2)]\lambda = n\lambda$.

ii) For $2 \leq i \leq n-1$,there are $n-i+3$ pair of degree 2 vertices and $n-i-1$ degree 4 vertices at a distance *i.*

$$\sum_{i=2}^{n-1}[(n-i+3)+(n-k-1)]\lambda^i = 2\sum_{i=2}^{n-1}(n-i+1)\lambda^i$$

iii) Lastly, there are 4 pair of degree 2 vertices at a distance 1 and no degree 4 vertices at a distance *n. i.e* $4\lambda^n$

combining the above expression we have the Acharya polynomial

$$AP(G,\lambda) = n\lambda + 2\sum_{i=2}^{n-1}(n-i+1)\lambda^i + 4\lambda^n$$

We know that $AI(G) = \frac{d}{d\lambda}AP(G,\lambda)$ at $\lambda = 1$

$$AI(G) = \frac{\partial}{\partial\lambda}AP(G,\lambda)$$

$$= \frac{\partial}{\partial\lambda}\left(n\lambda + 2\sum_{i=2}^{n-1}(n-i+1)\lambda^i + 4\lambda^n\right)$$

$$= n + 2\sum_{i=2}^{n-1}i(n-i+1)\lambda^{i-1} + 4n\lambda^{n-1}$$

At $\lambda = 1$

$$AI(G) = n + 2\sum_{i=2}^{n-1}i(n-i+1) + 4n$$

$$= 5n + 2\left[(n+1)\sum_{i=2}^{n-1}i - \sum_{i=2}^{n-1}i^2\right]$$

$$= \frac{n^3+3n^2+5n}{3}$$

Theorem 2: The Acharya polynomial of the para chain square cactus $Q_n(n \geq 2)$ is,

$$AP(G,\lambda) = 4\sum_{k=0}^{n-2}\lambda^{2k+1} + (6n-6)\lambda^2 + \sum_{k=2}^{n-1}(5n-5k-1)\lambda^{2k} + 4\lambda^{2n-1} + \lambda^{2n}$$

$$AI(G) = \frac{5n^3+9n^2+10n}{3}$$

Proof: Let x and y are any two vertices of $Q_n(n \geq 2)$, then the cactus graph $Q_n(n \geq 2)$ is biregular graph with degrees of vertices either 2 or 4. Let the distance between the vertices be $d(x,y)=i$,then we have following distances and the number of pair of vertices with these distances.

i) For $0 \leq k \leq n-2$, there are 4 pair of degree 2 vertices and no pair degree 4 vertices at a distance i where $i = 2k+1$, Thuthe term in polynomial is $4\sum_{k=0}^{n-2}\lambda^{2k+1}$

ii)There are $5(n-1)+1$ pair of degree 2 and $n-2$ pair of degree 4 vertices at a distance 2

i.e $\left[5(n-1)+1+(n-2)\right]\lambda^2 = (6n-6)\lambda^2$

iii) For $2 \leq k \leq n-1$, there are $4(n-k)$ pair of degree 2 vertices and $n-k-1$ pair degree 4 vertices at a distance i where $i = 2k$

i.e

$$\sum_{k=2}^{n-1}\left[4(n-k)+(n-k-1)\right]\lambda^{2k} = \sum_{k=2}^{n-1}(5n-5k-1)\lambda^{2k}$$

iv) There are 4 pair of degree 2 and no pair of degree 4 vertices at a distance $2n-1$
$4\lambda^{2n-1}$

v) There is 1 pair of degree 2 and no pair of degree 4 vertices at a distance $2n$
λ^{2n}

Combining all above distances we have Acharya polynomial as

$$AP(G,\lambda)=4\sum_{k=0}^{n-2}\lambda^{2k+1}+(6n-6)\lambda^2+\sum_{k=2}^{n-1}(5n-5k-1)\lambda^{2k}+4\lambda^{2n-1}+\lambda^{2n}$$

The Acharya index from the above expression is

$$AI(G)=\frac{\partial}{\partial\lambda}AP(G,\lambda)$$

$$=\frac{\partial}{\partial\lambda}\left(4\sum_{k=0}^{n-2}\lambda^{2k+1}+(6n-6)\lambda^2+\sum_{k=2}^{n-1}(5n-5k-1)\lambda^{2k}+4\lambda^{2n-1}+\lambda^{2n}\right)$$

$$=4\sum_{k=0}^{n-2}(2k+1)\lambda^{2k}+2(6n-6)\lambda+\sum_{k=2}^{n-1}(2k)(5n-5k-1)\lambda^{2k-1}+4(2n-1)\lambda^{2n-2}+(2n)\lambda^{2n}$$

At $\lambda=1$

$$AI(G)=4\sum_{k=0}^{n-2}(2k+1)+2(6n-6)+\sum_{k=2}^{n-1}(2k)(5n-5k-1)+4(2n-1)+(2n)$$

$$=22n-16+4\sum_{k=0}^{n-2}(2k+1)+\sum_{k=2}^{n-1}(2k)(5n-1)-10\sum_{k=2}^{n-1}k^2$$

$$=22n-16+8\sum_{k=0}^{n-2}k+4\sum_{k=0}^{n-2}1+(10n-2)\sum_{k=2}^{n-1}k-10\sum_{k=2}^{n-1}k^2$$

$$=\frac{5n^3+9n^2+10n}{3}$$

Theorem 3: The Acharya polynomial of the Ortho-chain square cactus $O_n(n\geq 5)$ is,

$$AP(G,\lambda)=2n\lambda+2n\lambda^2+(4n-3)\lambda^3+\sum_{i=4}^{n-1}(5n-5i+11)\lambda^i+13\lambda^n+6\lambda^{n+1}+\lambda^{n+2}$$

$$AI(G)=\frac{5}{6}n^3+\frac{11}{2}n^2+\frac{5}{3}n+3$$

Proof: Let x and y are any two vertices of $O_n(n\geq 5)$, then the cactus graph $O_n(n\geq 5)$ is biregular graph with degrees of vertices either 2 and 4 . Let the distance between the vertices be $d(x,y)=i$,then we have following distances and the number of pair of vertices with these distances.

i) There are $n+2$ pair of degree 2 and $n-2$ pair of degree 4 vertices at a distance 1, thus the first degree term is $\left[(n+2)+(n-2)\right]\lambda=2n\lambda$

ii) There are $n+3$ pair of degree 2 and $n-3$ pair of degree 4 vertices at a distance 2, thus the second term in the expression is $\left[(n+3)+(n-3)\right]\lambda^2=2n\lambda^2$.

iii) There are $3n$ pair of degree 2 and $n-4$ pair of degree 4 vertices at a distance 3, thus this term in the expression is $\left[3n+(n-3)\right]\lambda^3=(4n-3)\lambda^3$.

iv) For $4\leq i\leq n-1$, there are $4(n-i+3)$ pair of degree 2 vertices and $n-i-1$ pair degree 4 vertices at a distance i , Thus the term in polynomial is

$$\sum_{i=4}^{n-1}\left[(4n-4i+12)+n-i-1)\right]\lambda^i=\sum_{i=4}^{n-1}(5n-5i+11)\lambda^i$$

v) There are 13 pair of degree 2 vertices and no pair of degree 4 vertices at a distance *n*. *i.e*
$13\lambda^{n}$
vi) There are 6 pair of degree 2 vertices, at a distance and no pair of degree 4 vertices at a distance *n*.+1 *i.e*
$6\lambda^{n+1}$
*vii)*There is only 1 pair of degree 2 vertices at a distance *n*+2 *i.e*
$1\lambda^{n+2}$

Combining all the terms we have the Acharya polynomial as

$$AP(G,\lambda)=2n\lambda+2n\lambda^{2}+(4n-3)\lambda^{3}+\sum_{i=4}^{n-1}(5n-5i+11)\lambda^{i}+13\lambda^{n}+6\lambda^{n+1}+\lambda^{n+2}$$

$$AI(G)=\frac{\partial}{\partial\lambda}AP(G,\lambda)$$

$$=\frac{\partial}{\partial\lambda}\left(2n\lambda+2n\lambda^{2}+(4n-3)\lambda^{3}+\sum_{i=4}^{n-1}(5n-5i+11)\lambda^{i}+13\lambda^{n}+6\lambda^{n+1}+\lambda^{n+2}\right)$$

$$=2n+4n\lambda+3(4n-3)\lambda^{2}+\sum_{i=4}^{n-1}i(5n-5i+11)\lambda^{i-1}+13n\lambda^{n-1}+6(n+1)\lambda^{n}+(n+2)\lambda^{n+1}$$

At $\lambda=1$

$$AI(G)=2n+4n+3(4n-3)+\sum_{i=4}^{n-1}i(5n-5i+11)+13n+6(n+1)+(n+2)$$

$$=38n-1+(5n+11)\sum_{i=4}^{n-1}i-5\sum_{i=4}^{n-1}i^{2}$$

$$=\frac{5}{6}n^{3}+\frac{11}{2}n^{2}+\frac{5}{3}n+3$$

Theorem 5: The Acharya polynomial of the Para-chain hexagonal cactus graph $L_n(n\geq 3)$ is

$$AP(G,\lambda)=(2n+4)\lambda+6n\lambda^{2}+(11n-10)\lambda^{3}+4\sum_{k=1}^{n-2}(n-k+1)\lambda^{3k+1}+4\sum_{k=1}^{n-2}(n-k)\lambda^{3k+2}$$
$$+\sum_{k=2}^{n-1}(9n-9k-1)\lambda^{3k}+8\lambda^{3n-2}+4\lambda^{3n-1}+\lambda^{3n}$$

$$AI(G)=\frac{17n^{3}+21n^{2}+16n}{2}$$

Proof : Let x and y are any two vertices of $L_n(n\geq 3)$, then the cactus graph $L_n(n\geq 3)$ is biregular graph with degrees of vertices either 2 and 4 . Let the distance between the vertices be $d(x,y)=i$, for some distances the i is expressed in terms of *k*. We have following distances and the number of pair of vertices with these distances.

i) There are $2n+4$ pair of degree 2 vertices and no degree 4 vertices at a distance 1, thus the first term in expression is $(2n+4)\lambda$

ii) There are $6n$ pair of degree 2 vertices and no degree 4 vertices at a distance 2, thus the second term in expression is $6n\lambda^{2}$.

iii) There are $10(n-1)+2$ pair of degree 2 vertices and $n-2$ degree 4 vertices at a distance 3 , thus the third term in expression is $\left[(10(n-1)+2)+(n-2)\right]\lambda^{3}=(11n-10)\lambda^{3}$

iv) For $1\leq k\leq n-2$, there are $4(n-k+1)$ pair of degree 2 vertices and no degree 4 at a distance $3k+1$, thus $4\sum_{k=1}^{n-2}(n-k+1)\lambda^{3k+1}$

v) For $1 \le k \le n-2$ There are $4(n-k)$ pair of degree 2 vertices and no degree 4 at a distance $3k+2$, thus $4\sum_{k=1}^{n-2}(n-k)\lambda^{3k+2}$

vi) For $2 \le k \le n-1$ There are $8(n-k)$ pair of degree 2 vertices and $n-k-1$ degree 4 at a distance $3k$, thus $\sum_{k=2}^{n-1}\left[8(n-k)+(\mathrm{n-k-1})\right]\lambda^{3k} = \sum_{k=2}^{n-1}(9n-9k-1)\lambda^{3k}$

vii) There are 8 pair of degree 2 vertices and no degree 4 vertices at a distance $3n-2$, thus the term in expression is $8\lambda^{3n-2}$

viii) There are 4 pair of degree 2 vertices and no degree 4 vertices at a distance $3n-2$, thus the term in expression is $4\lambda^{3n-1}$

ix) There is only 1 pair of degree 2 vertices and no degree 4 vertices at a distance $3n$, thus the term in expression is λ^{3n}

combining all above terms we have the Acharya polynomial as

$$AP(G,\lambda) = (2n+4)\lambda + 6n\lambda^2 + (11n-10)\lambda^3 + 4\sum_{k=1}^{n-2}(n-k+1)\lambda^{3k+1} + 4\sum_{k=1}^{n-2}(n-k)\lambda^{3k+2} + \sum_{k=2}^{n-1}(9n-9k-1)\lambda^{3k} + 8\lambda^{3n-2} + 4\lambda^{3n-1} + \lambda^{3n}$$

$$AI(G) = \frac{\partial}{\partial\lambda}AP(G,\lambda)$$

$$= \frac{\partial}{\partial\lambda}\left(\begin{array}{c}(2n+4)\lambda + 6n\lambda^2 + (11n-10)\lambda^3 + 4\sum_{k=1}^{n-2}(n-k+1)\lambda^{3k+1} + 4\sum_{k=1}^{n-2}(n-k)\lambda^{3k+2} \\ + \sum_{k=2}^{n-1}(9n-9k-1)\lambda^{3k} + 8\lambda^{3n-2} + 4\lambda^{3n-1} + \lambda^{3n}\end{array}\right)$$

$$= (2n+4) + 12n\lambda + 3(11n-10)\lambda^2 + 4\sum_{k=1}^{n-2}(3k+1)(n-k+1)\lambda^{3k} + 4\sum_{k=1}^{n-2}(3k+2)(n-k)\lambda^{3k+1} + \sum_{k=2}^{n-1}(3k)(9n-9k-1)\lambda^{3k-1} + 8(3n-2)\lambda^{3n-1} + 4(3n-1)\lambda^{3n-2} + (3n)\lambda^{3n-1}$$

At $\lambda = 1$

$$AI(G) = (2n+4) + 12n + 3(11n-10) + 4\sum_{k=1}^{n-2}(3k+1)(n-k+1) + 4\sum_{k=1}^{n-2}(3k+2)(n-k) + \sum_{k=2}^{n-1}(3k)(9n-9k-1) + 8(3n-2) + 4(3n-1) + (3n)$$

$$= 86n - 46 + (12n+8)\sum_{k=1}^{n-2}k + (4n+4)\sum_{1}^{n-2}1 - 12\sum_{k=1}^{n-2}k^2$$

$$= \frac{17n^3 + 21n^2 + 16n}{2}$$

Theorem 6 The Acharya polynomial of the Para-chain hexagonal cactus graph $M_n (n \ge 4)$ is

$$AP(G,\lambda) = (2n+4)\lambda + (6n-2)\lambda^2 + (5n+2)\lambda^3 + (10n-14)\lambda^4 + \sum_{k=2}^{n-2}(6n-6k+8)\lambda^{2k+1} + \sum_{k=3}^{n-1}(11n-11k+8)\lambda^{2k} + 14\lambda^{2n-1} + 10\lambda^{2n} + 4\lambda^{2n+1} + 1\lambda^{2n+2}$$

$$AI(G) = \frac{17n^3}{3} + 19n^2 - \frac{5n}{3} + 4$$

Proof : Let x and y are any two vertices of $M_n(n \geq 4)$, then the cactus graph $M_n(n \geq 4)$ is biregular graph with degrees of vertices either 2 and 4 . Let the distance between the vertices be $d(x, y) = i$, for some distances the i is expressed in terms of k. We have following distances and the number of pair of vertices with these distances.

i) There are $2n+4$ pair of degree 2 vertices and no degree 4 vertices at a distance 1, thus the first term in expression is $(2n+4)\lambda$

ii) There are $5n$ pair of degree 2 vertices and $n-2$ degree 4 vertices at a distance 2, thus the second term in expression is $[5n+(n-2)]\lambda^2 = (6n-2)\lambda^2$.

iii) There are $5n+2$ pair of degree 2 vertices and no degree 4 vertices at a distance 3 , thus the third term in expression is $(5n+2)\lambda^3$.

iv) There are $9(n-1)-2$ pair of degree 2 vertices and $n-3$ degree 4 at a distance 4, thus $\left[(9(n-1)-2)+(n-3)\right]\lambda^4 = (10n-14)\lambda^4$

v) For $2 \leq k \leq n-2$ There are $6(n-k+1)+2$ pair of degree 2 vertices and no degree 4 at a distance $2k+1$, thus $\sum_{k=2}^{n-2}[6(n-k+1)+2]\lambda^{2k+1} = \sum_{k=2}^{n-2}(6n-6k+8)\lambda^{2k+1}$

vi) For $3 \leq k \leq n-1$, there are $10(n-k+1)-1$ pair of degree 2 vertices and $n-k-1$ degree 4 at a distance $2k$, thus $\sum_{k=3}^{n-1}\left[10(n-k+1)-1+(n-k-1)\right]\lambda^{2k} = \sum_{k=3}^{n-1}(11n-11k+8)\lambda^{2k}$

vii) There are 14 pair of degree 2 vertices and no degree 4 vertices at a distance $2n-1$, thus the term in expression is $14\lambda^{2n-1}$

viii) There are 10 pair of degree 2 vertices and no degree 4 vertices at a distance $2n$, thus the term in expression is $10\lambda^{2n}$

ix) There is only 4 pair of degree 2 vertices and no degree 4 vertices at a distance $2n+1$, thus the term in expression is $4\lambda^{2n+1}$

x) There is only 1 pair of degree 2 vertices and no degree 4 vertices at a distance $2n+2$, thus the term in expression is $1\lambda^{2n+2}$

Combining all above terms we have the Acharya polynomial as

$$AP(G,\lambda) = (2n+4)\lambda + (6n-2)\lambda^2 + (5n+2)\lambda^3 + (10n-14)\lambda^4 + \sum_{k=2}^{n-2}(6n-6k+8)\lambda^{2k+1} + \sum_{k=3}^{n-1}(11n-11k+8)\lambda^{2k} + 14\lambda^{2n-1} + 10\lambda^{2n} + 4\lambda^{2n+1} + 1\lambda^{2n+2}$$

$$AI(G) = \frac{\partial}{\partial\lambda} AP(G,\lambda)$$

$$= \frac{\partial}{\partial\lambda}\begin{pmatrix} (2n+4)\lambda + (6n-2)\lambda^2 + (5n+2)\lambda^3 + (10n-14)\lambda^4 + \sum_{k=2}^{n-2}(6n-6k+8)\lambda^{2k+1} \\ + \sum_{k=3}^{n-1}(11n-11k+8)\lambda^{2k} + 14\lambda^{2n-1} + 10\lambda^{2n} + 4\lambda^{2n+1} + 1\lambda^{2n+2} \end{pmatrix}$$

$$= (2n+4) + 2(6n-2)\lambda + 3(5n+2)\lambda^2 + 4(10n-14)\lambda^3 + \sum_{k=2}^{n-2}(6n-6k+8)(2k+1)\lambda^{2k}$$

$$+ \sum_{k=3}^{n-1}(11n-11k+8)(2k)\lambda^{2k-1} + 14(2n-1)\lambda^{2n-2} + 10(2n)\lambda^{2n-1} + 4(2n+1)\lambda^{2n} + (2n+2)\lambda^{2n+1}$$

At $\lambda = 1$

$$AI(G) = (2n+4) + 2(6n-2) + 3(5n+2) + 4(10n-14) + \sum_{k=2}^{n-2}(6n-6k+8)(2k+1)$$

$$+\sum_{k=3}^{n-1}(11n-11k+8)(2k) + 14(2n-1) + 10(2n) + 4(2n+1) + (2n+2)$$

$$= 2n+4+12n-4+15n+6+40n-56+(12n+10)\sum_{k=2}^{n-2}k - 12\sum_{k=2}^{n-2}k^2 + (6n+8)\sum_{k=2}^{n-2}1$$

$$+(22n+16)\sum_{k=3}^{n-1}k - 22\sum_{k=3}^{n-1}k^2$$

$$= \frac{17n^3}{3} + 19n^2 - \frac{5n}{3} + 4$$

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