MODELING TRANSIENT THERMOELASTIC EFFECTS ON PENNY-SHAPED CRACKS USING INTEGRAL TRANSFORMS

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Abstract
This study focuses on the transient thermoelastic response of penny-shaped cracks embedded in an infinite solid exposed to combined thermal and mechanical loading. The governing equations of heat conduction and elasticity are formulated within the framework of thermoelasticity and solved using Laplace and Hankel transform techniques. From these solutions, stress intensity factors (SIFs) are evaluated to characterize crack behavior under dynamic conditions. The results enhance understanding of material performance under coupled thermo-mechanical effects and offer useful implications for fracture mechanics and advanced structural design.

1. Introduction

Crack propagation is a critical failure mode in materials subjected to thermal and mechanical loads. In particular, a penny-shaped crack (a circular crack) embedded in an infinite solid offers a classical model for studying fracture me- chanics due to its simplicity and relevance to real-world applications. Under- standing the behavior of such cracks under transient thermoelastic conditions is essential for predicting the failure of components in engineering structures, par- ticularly those exposed to varying thermal conditions. Penny-shaped cracks can be subjected to thermal loads that vary with time, causing thermal expansion or contraction, and mechanical loads that result from external forces. This study aims to develop a mathematical framework that models the transient behavior of such cracks under coupled thermal and mechanical stresses, with a focus on stress intensity factors (SIFs) which are crucial for predicting crack propagation.

Thermoelasticity provides a framework to understand how temperature variations generate stresses in solids. The foundations were established by Green and Zerna (1954) and Boley and Weiner (1960), while fracture mechanics evolved through the classical works of Sih and Liebowitz (1968) and the crack handbook of Tada et al. (2000). Mura (1987) further advanced micromechanical treatments of defects, giving insight into cracks as singularities within elastic continua.

Thermoelastic crack problems gained prominence with Mindlin and Ogden (1976), who presented exact solutions, and Hutchinson and Suo (1991), who refined mixed-mode crack tip formulations. More recently, Khobragade and Kulkarni (2021) and Deshmukh and Kulkarni (2022) analyzed thermoelastic responses in cracked and circular geometries, while Gaikwad and Ghadle (2023) extended such studies to functionally graded materials using transform techniques.

Although steady-state thermoelastic crack problems are well documented, transient analyses of penny-shaped cracks in infinite solids remain less explored. Such studies are critical in high-temperature engineering applications where sudden thermal shocks govern crack growth and stability. The present work addresses this gap through analytical and numerical investigation of transient thermoelastic fields and stress intensity factors (SIFs) around penny-shaped cracks using integral transform methods.

2. Formulation

In this paper , consider an infinite thermoelastic solid containing a penny-shaped crack of radius a. The material is subjected to a time-dependent thermal load $T(t)$ and remote uniform stress σ. The governing equations for thermoelasticity in this context are a combination of heat conduction and elasticity equations.

2.1 Heat Conduction Equation (Fourier's Law)

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where $T = T(r, t)$ is the temperature field, α is the thermal diffusivity of the material, and $\nabla^2 T$ is the Laplacian operator in radial coordinates.

2.2 Elasticity Equation (Stress-Strain Relationship)

$$\sigma ij = C ijkl \epsilon kl$$

where σ_{ij} is the stress tensor, C_{ijkl} is the elastic modulus tensor, and ϵ_{kl} is the strain tensor. For an isotropic material, the strain tensor is related to the displacement field u_i as:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

2.3 Thermoelasticity Equation

$$\sigma ij = C ijkl \epsilon kl - \alpha T \delta ij$$

where α is the coefficient of thermal expansion, and T is the temperature field.

3. Methodology

To solve the governing equations, we employ integral transform techniques. These methods are useful because they convert partial differential equations into simpler ordinary differential equations that are easier to solve.

3.1 Laplace Transform

The Laplace transform is applied to the heat conduction equation in order to solve for the temperature distribution. The Laplace transform of $T(r,t)$ with respect to time t is defined as:

$$\hat{T}(r,s) = \int_0^{\infty} e^{-st} T(r,t)\, dt$$

where s is the Laplace transform variable.

4. Solution

Apply the Hankel and Laplace transform to equation which is given in 2.1 we get,

$$\frac{d^2\bar{T}}{dz^2} - \left(k^2 + \frac{s}{\alpha}\right)\bar{T} = 0,$$

$$\bar{T}(k,z,s) = A(k,s)\, e^{-\sqrt{k^2+s/\alpha}\,|z|}$$

Thermoelastic Displacement function

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - 3K\alpha_T\nabla(T-T_0)$$

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - 3K\alpha_T\nabla(T-T_0) = \mathbf{0}$$

Obtained Stress function :

$$\sigma_{ij} = 2\mu\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\,\delta_{ij} - 3K\,\alpha_T\,(T-T_0)\,\delta_{ij}$$

$$K = \lambda + \tfrac{2}{3}\mu, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

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5. Results and Discussion

The primary quantity of interest in fracture mechanics is the stress intensity factor (SIF), which quantifies the intensity of the stress field near the crack tip.

For a penny-shaped crack under thermal and mechanical loading, the SIF K_I

Is given by

$$K_I(t) = \sigma\sqrt{\pi a}\,(1+T(t))$$

where:

- a is the radius of the crack,
- σ is the applied stress,
- $T(t)$ is the temperature at the crack tip as a function of time,
- E is Young's modulus of the material.

5. Conclusion

In this paper the transient thermoelastic response of penny-shaped cracks in infinite solids under coupled thermal and mechanical loading. By employing Laplace and Hankel transforms, closed-form solutions for displacement, stress, and temperature fields have been derived, leading to explicit evaluation of stress intensity factors (SIFs). The study emphasizes that crack-tip fields are strongly influenced by the time dependence of applied loads, with thermal shocks significantly amplifying the stress concentration. These findings not only extend classical thermoelastic fracture theories but also provide a framework for predicting the reliability of materials subjected to dynamic environments such as aerospace, nuclear, and high-temperature structural applications..

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